1. Suppose that r indistinguishable balls are placed in n distinguishable boxes so that each distinguishable arrangement is equally likely. Find the probability that no box will be empty.

   Solution: Suppose, $r \geq n$. Let A be the event that no box will remain empty and B be the total possible no of ways that r indistinguishable balls are placed in n distinguishable boxes. Now consider the equation,
$$a_1 + a_2 + \cdots + a_n = r,$$
where a_i corresponds the cumber of balls in i-th box, for $i = 1, \cdots, n$. Hence
$$|A| = \binom{r-1}{n-1},$$
i.e. the number of solutions of the above equation in positive integers and
$$|B| = \binom{r+n-1}{n-1},$$
i.e. the number of solutions of the above equation in non-negative integers. Therefore, the required probability is
$$P = \frac{\binom{r-1}{n-1}}{\binom{r+n-1}{n-1}}.$$
□

2. A box contains 6 white balls and 9 black balls. A sample of 5 balls is drawn at random without replacement. Let A_3 denote the event that the ball drawn on the 3rd draw is white. Let B_1 denote the event that the sample of 5 balls contains exactly 1 white ball. Find $P(A_3|B_1)$.

Solution: Now, $A_3 \cap B_1 = \{B, B, W, B, B\}$, where B denotes the black ball and W denotes the white ball. Hence
$$P(A_3 \cap B_1) = \frac{9}{15} \cdot \frac{8}{14} \cdot \frac{6}{13} \cdot \frac{7}{12} \cdot \frac{6}{11}.$$
Then $B_1 = \{W, B, B, B, B\} \cup \{B, W, B, B, B\} \cup \{B, B, W, B, B\} \cup \{B, B, B, W, B\} \cup \{B, B, B, B, W\}$. Therefore,
$$P(A_3|B_1) = \frac{P(A_3 \cap B_1)}{P(B_1)} = \frac{1}{5}.$$
□

3. Let $X \sim Bernoulli(p)$ and $Y \sim Bernoulli(q)$ be independent. Find the distribution of $Z = X+Y-XY$ and conditional distribution of $X|Z = 1$.

Solution: Given $P(X = 1) = p$, $P(X = 0) = 1 - p$, $P(Y = 1) = q$ and $P(Y = 0) = 1 - q$. Now, $Z = \{0, 1\}$. Hence

$$\begin{aligned}
&P(Z = 0) \\
&= P(X + Y - XY = 0) \\
&= P(X = 0, Y = 0) \\
&= P(X = 0)P(Y = 0) \\
&= (1 - p)(1 - q) \\
&= 1 - (p + q - pq).
\end{aligned}$$

Again

$$\begin{aligned}
&P(Z = 1) \\
&= P(X + Y - XY = 1) \\
&= P(\{X = 0, Y = 1\} \cup \{X = 1, Y = 0\} \cup \{X = 1, Y = 1\}) \\
&= P(X = 0)P(Y = 1) + P(X = 1)P(Y = 0) + P(X = 1)P(Y = 1) \\
&= (1 - p)q + p(1 - q) + pq \\
&= p + q - pq.
\end{aligned}$$

Therefore, $Z \sim Bernoulli(p + q - pq)$. Now,

$$\begin{aligned}
&P(X|Z = 1) \\
&= \frac{P(X = 0, Z = 1)}{P(Z = 1)} \\
&= \frac{P(X = 0, Y = 1)}{p + q - pq} \\
&= \frac{P(X = 0)P(Y = 1)}{p + q - pq} \\
&= \frac{(1 - p)q}{p + q - pq}.
\end{aligned}$$

Again,

$$\begin{aligned}
&P(X|Z = 1) \\
&= \frac{P(X = 1, Z = 1)}{P(Z = 1)} \\
&= \frac{P(X = 1, Y = 1) + P(X = 1, Y = 0)}{p + q - pq} \\
&= \frac{P(X = 1)P(Y = 1) + P(X = 1)P(Y = 0)}{p + q - pq} \\
&= \frac{pq + p(1 - q)}{p + q - pq} \\
&= \frac{p}{p + q - pq}.
\end{aligned}$$

Therefore, $X|Z = 1 \sim Bernoulli(\frac{p}{p+q-pq})$. □

4. There are N students in the Probability class. Of them, F are female, P of them use a pencil (instead of a pen), and G of them are wearing eye glasses. A student is chosen at random from the class. Define the following events:
A_1={the student is a female}
A_2={the student uses a pencil}
A_3={the student is wearing eye glasses}
(a) Let $N = 150,\ F = 90,\ P = 60,\ G = 30$. Show that it is impossible for these events to be mutually independent.
(b) Is there an example of (N, F, P, G) where the above events are pairwise independent?

Solution: (a) Let, the events are mutually independent.

$$\begin{aligned}
&P(A_1 \cup A_2 \cup A_3) \\
&= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3) \\
&= P(A_1) + P(A_2) + P(A_3) - P(A_1)P(A_2) - P(A_2)P(A_3) - P(A_3)P(A_1) + P(A_1 \cap A_2 \cap A_3) \\
&= \frac{19}{25} + P(A_1 \cap A_2 \cap A_3).
\end{aligned}$$

Therefore

$$P(A_1 \cap A_2 \cap A_3) \leq \frac{6}{25} = \frac{36}{150}.$$

Hence, the proof.
(b) consider the example- $N = 4,\ F = 2,\ G = 2,\ F \cap G = 1,\ P = 0.$ $\square$